ON THE FURUTSU-NOVIKOV-DONSKER FORMULA

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ABSTRACT. Recently often used Novikov's formula can be understood as a special property of Gaussian path integral (or free field) and generalized to partial integral concept well known in the context of constructive field theory[Glimm-Jaffe]. Then we move on to non Gaussian case which is not so famous in classical field theoretical literature.

1. GAUSSIAN CASE

[Novikov] introduced a following expression:

$$\langle f_i(\vec{x},t)R[f]\rangle = \int F_{ik}(\vec{x}-\vec{x}')\left\langle \frac{\delta R[f]}{\delta f_k(\vec{x}',t)d^3x'dt}\right\rangle d^3x'.$$

Foot indices or dimensionality of the space is unimportant for the proof. We must focus on the foundation of special cancellation occurs in OPE for the free field. Generalization is as follows.

Lemma 1.1. Decomposition of the Normal Ordered Product

$$\phi(x):\phi^n(y):=:\phi(x)\phi^n(y):+nG(x,y):\phi^{n-1}(y):$$

which can be proved using OPE.

Lemma 1.2. Integration by Parts

$$\langle\phi(x)A[\phi]\rangle = \int dy G(x,y)\left\langle\frac{\delta A[\phi]}{\delta\phi(y)}\right\rangle$$

Theorem 1.3. Generalized Novikov Identity

$$\langle:\phi^n(x):R[\phi]\rangle = \int dy G(x,y)\left\langle:\phi^{n-1}(x):\frac{\delta R}{\delta\phi(y)}\right\rangle$$

Date: August 8, 2008.

Proof. For a free field, OPE naturally exhibits its power. Using above Lemmas, we obtain the result.

$$\begin{aligned}
\langle : \phi^n(x) : R \rangle &= \left\langle \phi(x) : \phi^{n-1}(x) : R \right\rangle - (n-1) G(x,x) \left\langle : \phi^{n-2}(x) : R \right\rangle \\
&= \int dy G(x,y) \left\langle \frac{\delta}{\delta\phi(y)} : \phi^{n-1}(x) : R \right\rangle - (n-1) G(x,x) \left\langle : \phi^{n-2}(x) : R \right\rangle \\
&= \int dy G(x,y) \left\langle : \phi^{n-1}(x) : \frac{\delta R}{\delta\phi(y)} \right\rangle
\end{aligned}$$

□

2. Non Gaussian Case

Define non Gaussian measure:

$$d\mu_V \stackrel{\text{def}}{=} D\phi e^{-V[\phi]} / \int D\phi e^{-V[\phi]}$$

where $V[\phi] = \int dx : \mathrm{Poly}[\phi] :$

Theorem 2.1. Generalized IP

$$\langle \phi(x) A[\phi] \rangle_V = \int dy G(x,y) \left\langle \frac{\delta A[\phi]}{\delta\phi(y)} - A \frac{\delta V}{\delta\phi(y)} \right\rangle_V$$

References

[Novikov] E.A. Novikov, *Functionals and the Random-Force Method in Turbulence Theory*, JETP, **47**, 1919 (1964).

[Glimm-Jaffe] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, Springer-Verlag, 1987.

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